J. Fluid Mech. (1990), *vol.* 221, *pp.* 675–683
Printed in Great Britain

The sound field due to an oscillating bubble near an indented free surface

By MICHAEL S. LONGUET-HIGGINS
Center for Studies of Nonlinear Dynamics, La Jolla Institute, PO Box 1434, La Jolla, CA 92038, USA

(Received 5 October 1989 and in revised form 21 May 1990)

The problem is solved of the flow due to a rapidly oscillating source situated near a plane surface with a hemispherical indentation. The strength of the far-field flow is evaluated, and also the natural frequency of oscillation of a small bubble centred at the same position as the source.

1. Introduction

It has recently been demonstrated both by laboratory experiments (Pumphrey & Crum 1989) and by numerical calculations (Oguz & Prosperetti 1990*a*) that the main source of underwater sound from raindrops arises from the detachment of a bubble from the lowest point of the small indentation which is caused by the impact. The diameter of the oscillating bubble is generally much smaller than that of the surface indentation (see figure 1) and is also less than the depth of the bubble below the indentation. All these distances are generally small compared to the wavelength of the sound emitted by the bubble, whose frequency, of order 15 kHz, corresponds to a sonic wavelength of order 10 cm.

If the surface were simply plane, then the acoustical field at a distance that is large compared to the wavelength, would be simply that of a dipole. The questions that we wish to consider are these: How does the presence of the indentation affect the strength of the radiated sound, and also the natural frequency of oscillation of the bubble?

In view of the lengthscales mentioned above it is permissible to adopt an approximation usual in such problems, namely that the flow is locally incompressible. Moreover, since the timescale for the evolution of the cavity is of the order of several milliseconds, which is many times the period of oscillation of the bubble, the cavity may be assumed to be of constant shape during the ringing of the bubble. Associated with this fact is that the forces governing the oscillation of the bubble (surface tension and internal pressure) are many times the forces (mainly inertial, but also gravity and surface tension) tending to change the shape of the cavity.

Further we note that the detachment of the bubble from the free surface is very rapid, on a timescale comparable to an oscillation period. It then 'rings' at a relatively steady distance below the cavity. Bjerknes forces, tending to separate the bubble further from the free surface, are proportional to the square of the bubble perturbation, and so quickly become negligible.

Under these circumstances it is possible to treat the later, linear stages of the bubble oscillation as a problem of potential flow with a fixed configuration of the free surface. For simplicity we shall assume this to be a hemisphere, which is not far from

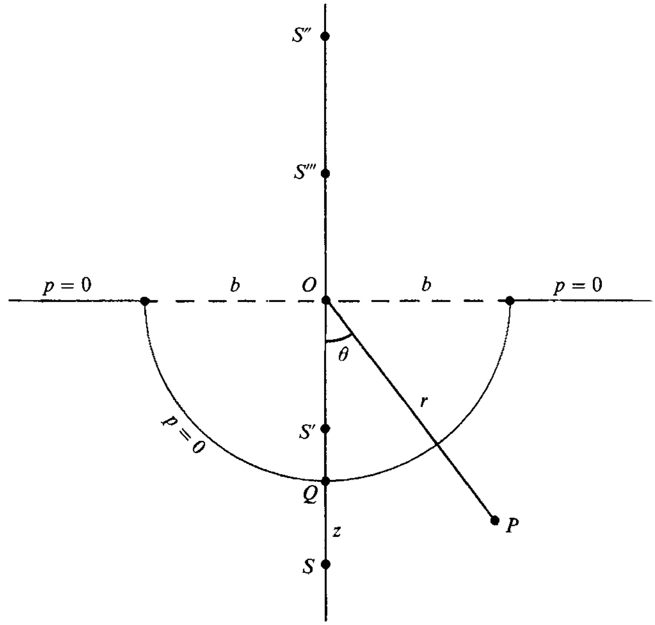

FIGURE 1. Definition diagram for lengths and coordinates.

the observed configuration at a short time after release of the bubble (Pumphrey & Crum 1989). The problem thus reduces to that shown in figure 1, with a high-frequency source S situated below a pressure-release surface in the form of a plane having a hemispherical identification.

In this note we shall solve the general problem very simply by the method of images. We obtain both the strength of the far field and the oscillation frequency as functions of two independent parameters. One simple result is that when the bubble is close to the bottom of the cavity, the strength of the dipole field is just 3 times what it would be if the bubble were at the same distance below a simple plane surface. This result has been applied directly to an analytic model of noise from raindrops (Longuet-Higgins 1990).

During preparation of this paper, we became aware of a somewhat similar investigation (Oguz & Prosperetti 1990*b*) in which the indentation of the surface has a more complicated analytic form. Their solution is expressed in terms of an infinite series. In our model, the solution is obtained in closed form, with a correspondingly simpler interpretation. In the Appendix to this paper we compare the oscillation frequencies calculated from the two different models and find a satisfactory agreement.

2. Evaluation of the far field

Figure 1 shows the axial plane section of a hemispherical indentation, centre O and radius b, in an otherwise plane horizontal surface. A pulsating source of radian frequency ω lies at a point S on the vertical axis, at a depth z below the lowest point Q on the sphere. We denote by r and θ the polar coordinates of a typical point P in the fluid, with respect to the origin O.

The lengths b and z are assumed to be small compared to the wavelength $2\pi c/\omega$ of the radiation from the source (c being the sound speed) so that the fluid in the near field may be assumed incompressible. Moreover, the deformation of the free surface under gravity and surface tension is assumed slow compared to ω. Our problem then reduces to finding a velocity potential ϕ behaving like a source

$$\phi_S = \frac{A}{PS}\cos\omega t \tag{2.1}$$

near S and satisfying the boundary condition

$$\frac{\partial\phi}{\partial t} = 0 \tag{2.2}$$

both on the plane boundary $\theta = \frac{1}{2}\pi$ and on the hemisphere $r = b$, $|\theta| \leqslant \frac{1}{2}\pi$.

The solution is obtained by first placing a sink

$$\phi'_S = -\frac{B}{PS'}\cos\omega t \tag{2.3}$$

at the image point S' where

$$OS' = \frac{b^2}{b+z} \tag{2.4}$$

and

$$B = \frac{b}{b+z}A. \tag{2.5}$$

This makes p vanish on the hemisphere. We then place a sink and source at the points S'' and S''' which are images of S and S' in the plane boundary. This makes p vanish both on the hemisphere and on the plane. The total potential, near the hemisphere, is thus

$$\phi = \left[A\left(\frac{1}{PS}-\frac{1}{PS''}\right)-B\left(\frac{1}{PS'}-\frac{1}{PS'''}\right)\right]\cos\omega t. \tag{2.6}$$

Now in the far field, the source and sink at S and S'' can be approximated in the usual way by the dipole

$$\frac{A\omega}{c}SS'\frac{\cos\theta}{r}\sin\omega\left(t-\frac{r}{c}\right) \tag{2.7}$$

and similarly for the sink and source at S' and S'''. As a result, the total potential in the far field is given by

$$\phi \sim \frac{2\omega}{c}\left[A(b+z)-\frac{Bb^2}{b+z}\right]\frac{\cos\theta}{r}\sin\omega\left(t-\frac{r}{c}\right). \tag{2.8}$$

On substituting the value of B from (2.5) this expression becomes

$$\phi \sim \frac{2\omega b}{c}AF\frac{\cos\theta}{r}\sin\omega\left(t-\frac{r}{c}\right) \tag{2.9}$$

where

$$F = \frac{b+z}{b}-\frac{b^2}{(b+z)^2}. \tag{2.10}$$

Hence

$$F = \eta G(\eta), \quad \eta = z/b, \tag{2.11}$$

where

$$G(\eta) = \frac{(1+\eta)^3-1}{\eta(1+\eta)^2} = \frac{3+3\eta+\eta^2}{1+2\eta+\eta^2}. \tag{2.12}$$

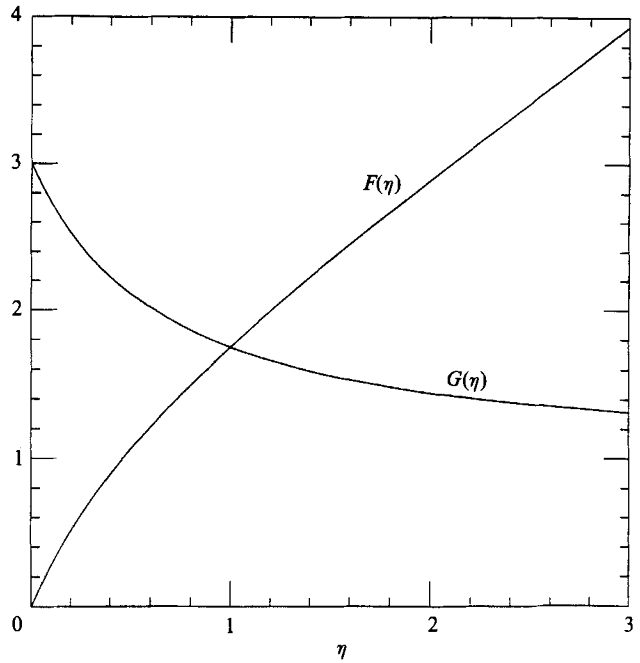

FIGURE 2. Graph of $F(\eta)$, giving the strength of the dipole field as a function of $\eta = z/b$; see equation (2.9).

Note that in the absence of the hemisphere, that is if $b \ll z$ or $\eta \gg 1$, then $G \sim 1$ and (2.9) reduces to

$$\phi \sim \frac{2\omega z}{c} A \frac{\cos\theta}{r} \sin\omega\left(t - \frac{r}{c}\right), \tag{2.13}$$

the ordinary dipole field for a source in the neighbourhood of a plane free surface. On the other hand in the presence of the hemisphere, the potential (2.13) is multiplied by the factor $G(\eta)$, which is greater than 1. For example, when $\eta \ll 1$ or $z \ll b$ we have $G \sim 3$ and the amplitude of the sound is three times that in the absence of the hemisphere.

Physically, we may say that instead of the boundary of the fluid being on the horizontal plane through Q (in which case the pressure-release surface would be at a uniform height z above S), in general the free surface lies above this plane. Hence there is an additional mass of fluid acting to constrain the fluid motion across the plane.

Figure 2 is a graph of $F(\eta)$, showing how the strength of the far field (2.9) increases when b is fixed and z increases from zero.

3. Frequency of the source

Assuming that the radius a of the bubble is small compared to both b and z we may find the frequency ω of oscillation by equating the mean kinetic energy of the fluid motion to the mean potential energy of the gas in the bubble (cf. Minnaert 1933). The mean potential energy of the gas is easily found to be given by

$$E_{\mathrm{P}} = 3\pi\gamma p_0 A^2/a^3\omega^2, \tag{3.1}$$

where γ is the ratio of the specific heats ($= 7/5$) and p_0 is the equilibrium pressure in the bubble. The kinetic energy can be evaluated from the expression

$$E_K = \tfrac{1}{2}\rho \iint_\Sigma \phi \nabla\phi \cdot \boldsymbol{n}\,\mathrm{d}\Sigma, \tag{3.2}$$

where ρ denotes the density and $\boldsymbol{n}$ is the outward normal to the boundary Σ of the fluid. For small values of a/b and a/z the only significant contribution to (3.2) comes from the surface Σ_B of the bubble and we find

$$E_K = \tfrac{1}{2}\rho \iint_{\Sigma_B} (\phi_0+\phi_1) \nabla\phi_0 \cdot \boldsymbol{n}\,\mathrm{d}\Sigma_B. \tag{3.3}$$

Here ϕ_0 is the singular potential (2.1) and ϕ_1 denotes the remaining terms in the complete potential (2.6). To a sufficient approximation the potential ϕ_1 can be evaluated at the point S, that is in equation (2.6) we may take P at S. Now from figure 1 we have

$$\left.\begin{aligned} SS'' &= 2(b+z), \\ SS' &= (b+z)-\frac{b^2}{b+z} = \frac{(b+z)^2-b^2}{b+z}, \\ SS''' &= (b+z)+\frac{b^2}{b+z} = \frac{(b+z)^2+b^2}{b+z}. \end{aligned}\right\} \tag{3.4}$$

Hence near S we have

$$\phi \sim A\left[\frac{1}{r}-\frac{1}{2(b+z)}\right] - B\left[\frac{(b+z)}{(b+z)^2-b^2}-\frac{(b+z)}{(b+z)^2+b^2}\right]. \tag{3.5}$$

On substituting B from (2.5), and simplifying, we obtain

$$\phi \sim \frac{A}{r}-\frac{A}{2z}H\left(\frac{z}{b}\right) \tag{3.6}$$

where
$$H(\eta) = \frac{\{(1+\eta)^4-1\}+4(1+\eta)}{\{(1+\eta)^4-1\}(1+\eta)}\eta \tag{3.7}$$

$$= \frac{4+8\eta+6\eta^2+4\eta^3+\eta^4}{4+10\eta+10\eta^2+5\eta^3+\eta^4}. \tag{3.8}$$

From (3.2) and (3.6) we have

$$E_K = \frac{\pi\rho A^2}{a}\left[1-\frac{a}{2z}H(\eta)\right] \tag{3.9}$$

and on equating this expression to (3.1) we find

$$\omega^2 = \frac{3\gamma p_0}{\rho a^2}\left[1-\frac{a}{2z}H(\eta)\right]^{-1}. \tag{3.10}$$

So, to lowest order in a/z there is a proportional change in frequency $\omega^{-1}\Delta\omega$ given by

$$\frac{\Delta\omega}{\omega} = \frac{a}{4z}H(\eta). \tag{3.11}$$

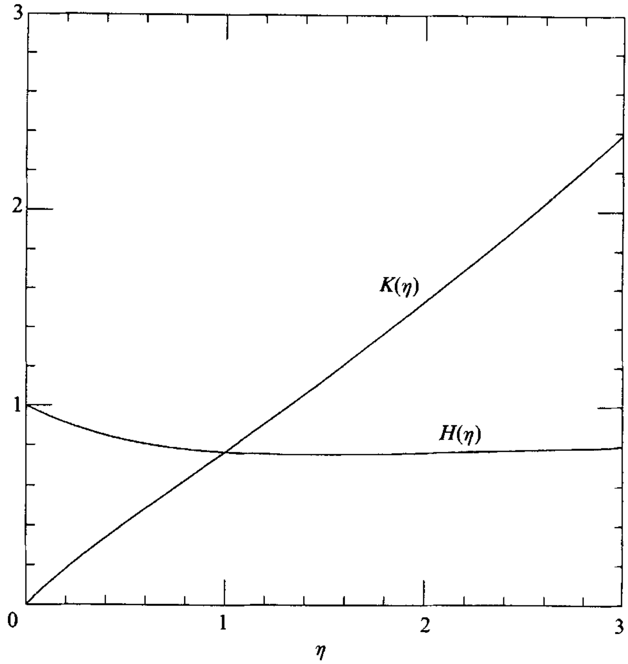

FIGURE 3. Graph of $K(\eta)$, giving the proportional increase in frequency as a function of $\eta = z/b$; see equation (3.13).

For example, when the hemispherical indentation is relatively small ($b \ll z$, $\eta \gg 1$), then $H \sim 1$ and we retrieve the known formula

$$\frac{\Delta\omega}{\omega} = \frac{a}{4z}. \tag{3.12}$$

On the other hand when the bubble is relatively close to the bottom of the indentation, i.e. $\eta \ll 1$, we have $H \sim 1$, so that (3.12) applies in that case also. Generally, (3.8) shows that $H(\eta) < 1$, hence the change in frequency of the bubble is always less than that given by the plane formula (3.12). On the other hand it is always greater than $a/4(b+z)$, which would be the value for a bubble at depth $(b+z)$ in the absence of an indentation.

For a fixed indentation radius b but variable z it is convenient to write

$$\frac{\Delta\omega}{\omega} = \frac{a}{4b}K(\eta), \tag{3.13}$$

where

$$K(\eta) = \eta H(\eta). \tag{3.14}$$

The function $K(\eta)$ is shown plotted against η in figure 3.

4. Discussion and conclusions

We have shown that the strength of the far field from an oscillating source lying just beneath a hemispherical indentation exceeds that in the corresponding case of a simpler plane by a factor of between 1 and 3. The change in the natural frequency of oscillation of a bubble due to its proximity to the hemispherical indentation is always less than that in the corresponding plane situation.

The theory that we have given is of course linear in the amplitude ϵa of the bubble oscillation. All terms of order ϵ^2, which will include the Bjerknes force tending to increase the distance between the bubble and the indented free surface, have here been neglected. For a bubble originating at the free surface as in the experiments of Pumphrey & Crum (1989), the ratio ϵ may well be of order 1 initially. But owing to damping of the oscillation (which we have also neglected) there may be at least some time interval during which ϵ is small enough for the linear theory to apply. The experiments show that these are circumstances in which a local frequency and amplitude of oscillation are well defined. A complete theory for the detachment of a bubble from a free surface has still to be given.

The work in this paper has been supported by the office of Naval Research under Contract Number N00014-88C-0563.

Appendix. Comparisons with Oguz & Prosperetti (1990*b*)

It is interesting to compare our results for a hemispherical indentation with those for the analytic model employed by Oguz & Prosperetti (1990*b*); see figure 4. To express the depression ζ of each point on the curves in figure 4 analytically as a function of its distance ϖ from the axis would involve some complicated transformations. However, one can easily see that each curve is in fact closely fitted by a Gaussian expression

$$\zeta = \alpha\, \mathrm{e}^{-\varpi^2/2\beta^2}, \tag{A 1}$$

where α, β are constants. In figure 4 we have indicated the width W of each curve at half the maximum depression. W is related to β by

$$W/\beta = (2\ln 2)^{\frac{1}{2}} = 1.774. \tag{A 2}$$

It will be seen that W, hence β, is almost exactly the same for each of the curves.

To compare each indentation in figure 4 with an equivalent hemisphere, we shall equate its volume V, given by

$$V = \int_0^{\alpha} \pi\varpi^2\, \mathrm{d}\zeta = 2\pi\alpha\beta^2, \tag{A 3}$$

to the volume of a hemispherical indentation of radius b, that is

$$V = \tfrac{2}{3}\pi b^3. \tag{A 4}$$

Hence we take

$$b = (3\alpha\beta^2)^{\frac{1}{3}}. \tag{A 5}$$

We shall also take z to be the depth of the bubble's centre below the lowest point of the indentation in figure 4. Choosing the unit of length so that the depth h of the bubble's centre below the plane surface is unity, the bubble's radius is then 0.1, in figure 4. Graphically we also find that $W = 0.379$, hence

$$\beta = 0.322 \tag{A 6}$$

from (A 2). From these numbers we derive the entries in table 1.

The ratio of the frequency of oscillation ω to the frequency ω_0 in an unbounded fluid, according to (3.10) is given by

$$\frac{\omega^2}{\omega_0^2} = \left[1 - \frac{a}{2z} H(\eta)\right]^{-1}. \tag{A 7}$$

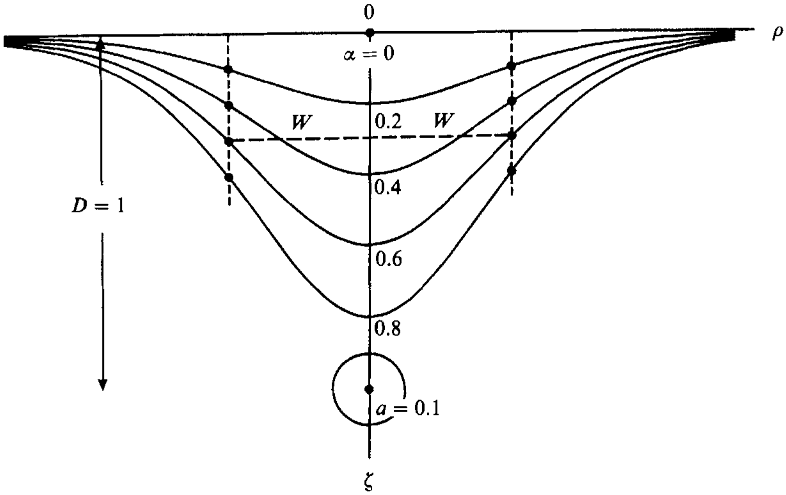

FIGURE 4. (Adapted from Oguz & Prosperetti, 1990*b*.) Axial cross-sections of the indented plane surface when $a = 0.1$ and $\alpha = 0, 0.2, 0.4, 0.6, 0.8$.

						ω^2/ω_0^2	
α	b	z	η	$H(\eta)$	$\left[1-\frac{a}{2z}H(\eta)\right]$	$[\ ]^{-1}$	Oguz & Prosperetti (1990*b*) figure 7
0	0	1.0	∞	1.000	0.950	1.053	1.051
0.2	0.396	0.8	2.019	0.767	0.952	1.050	1.056
0.4	0.499	0.6	1.202	0.759	0.937	1.068	1.067
0.6	0.571	0.4	0.700	0.793	0.901	1.110	1.094
0.8	0.629	0.2	0.318	0.872	0.782	1.279	1.220

TABLE 1. Calculation of ω^2/ω_0^2 for a hemispherical indentation roughly equivalent to the model of figure 4

This is shown in the next-to-last column of table 1. In the last column is shown the frequency as calculated by Oguz & Prosperetti, their figure 7. The agreement is remarkably good. As might be expected, the closest agreement is for $\alpha = 0.4$, when W and α, representing the horizontal and vertical scales of the indentation, are most nearly equal.

We may conclude that for indentations of a reasonable shape the frequency of oscillation can be calculated roughly from knowing (1) the total volume of the indentation, (2) the bubble's radius and (3) its distance below the bottom of the indentation. Moreover, for such a calculation the hemispherical model, which involves simple analytic expressions only, can be quite adequate.

REFERENCES

LONGUET-HIGGINS, M. S. 1990 An analytic model of sound production by raindrops. *J. Fluid Mech.* **214**, 395–410.

MINNAERT, M. 1933 On musical air-bubbles and the sounds of running water. *Phil. Mag.* **16**, 235–248.

Oguz, H. N. & Prosperetti, A. 1990*a* Bubble entrainment by the impact of drops on liquid surfaces. *J. Fluid Mech.* **219**, 143–179.

Oguz, H. N. & Prosperetti, A. 1990*b* Bubble oscillations in the vicinity of a nearby plane free surface. *J. Acoust. Soc. Am.* (submitted).

Pumphrey, H. C. & Crum, L. A. 1989 Sources of ambient noise in the ocean: an experimental investigation. *University of Mississippi, Tech. Rep.* NCPA LC.01 1989. 96 pp.